Listing of the Summaries of the

Rockefeller Mathematics Seminar Spring 2009

Organized by: J.J.P. Veerman**), M.J. Feigenbaum, F.M. Tangerman.

Summaries by J. J. P. Veerman and F. M. Tangerman

in collaboration with — and with advice from — the speakers.

Special thanks to:
Melanie Lee,
for her efficient help in the organization and dissemination of this seminar.

**) e-mail: veerman@pdx.edu

Universality in Dissipative Henon Maps

Marco Martens, SUNY Stony Brook,

email: marco@math.sunysb.edu.

May 11, 2pm

In a family of smooth unimodal maps of the unit interval I_0, such as $f : x \to ax(1-x)$, we choose a value a_0 of the parameter a based on a topological prescription. It follows from that description that at parameter value f has an unstable 2^n-periodic orbit for each non-negative integer n. The limit (as $n \to \infty$) of these orbits is an attracting attracting Cantor set O_f.

The topological (or combinatorial) prescription to choose a_0 is as follows. There is an interval of a's such that f has an attracting set consisting of two disjoint intervals I_1 and I_1', and f maps these intervals onto one another. The renormalization operator $\mathcal{R} : f \to \mathcal{R}f$ can then be defined as $f^2 = f \circ f$ on the interval I_1 rescaled to $[0,1]$. There is now a subinterval of parameters a such that $\mathcal{R}f$ 'looks like' f. More precisely it has an attracting set consisting of of two joint intervals I_2 and I_2' so that the renormalization can be applied again. The value a_0 is now uniquely determined by the requirement that f can be renormalized infinitely often (f is said to be *infinitely renormalizable.*

Fix $a = a_0$. The two most important immediate consequences of this construction are the existence of unstable 2^n-periodic orbits alluded to above, and the existence of "scalings". After renormalizing once, the positions of the intervals I_1 and I_1' relative to one another are characterized by two real numbers in $(0,1)$ (scalings): Let J_1 be the smallest interval containing I_1 and I_1', then $\sigma_1 = |I_1|/|J_1|$ and similarly for σ_1'. After renormalizing twice we get 4 scalings, namely two for those characterizing the relative positions of I_2 and I_2', and another two for the positions of their images under f. At level n of the renormalization we obtain 2^n scalings. The procedure describes the dynamics on ever smaller scales.

In this setting $\mathcal{R}^n f$ converges to a smooth map f_* with its own attracting Cantor set O_{f_*}. Even more importantly $f|_{O_f}$ and $f|_{O_{f_*}}$ are topologically conjugate. Now something subtle happens. One can define scalings for both f and f_* independently. It turns out that the deep level scalings of f converge to the corresponding ones for f_*. Thus the topological information that determined also fixes the *values* of scalings, and determines the local geometry of the invariant Cantor set O_f. This is called *rigidity* (topology determines geometry). This can also be expressed as follows: f and f_* are topologically conjugated on the attracting Cantor set, and this conjugation is $C^{1+\alpha}$ (ie: is differentiable and its derivative is α-Hölder).

Next let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a strongly dissipative Henon map, that is: $F : (x,y) \to (f(x) - \epsilon(x,y), x)$, where $0 < |\epsilon| << 1$ (strong dissipation), and f is a unimodal map as before (the image is in a small neighborhood of the parabola $(f(x), x)$). So F is a 'thickened' unimodal map. As before, for certain parameter values there is an attracting set consisting of 2 disjoint simply connected domains S_1 and and S_1', which are exchanged by the F. S_1 contains the so-called "tip" of the invariant Cantor set O_F (*akin* to the critical value). The renormalization $\mathcal{R}F$ of F is defined as F^2 on S_1 (nonlinearly) rescaled to the unit square and of Henon form. The requirement that F is infinitely renormalizable uniquely determines a parameter value, say a_0. Also in this case the construction implies the existence of periodic orbits converging to the invariant Cantor set, and scalings can be defined.

Theorem 1: *(Collet, Eckmann, Koch) Successive renormalizations converge to a unique ('universality') fixed point:*

$$\mathcal{R}^n F \longrightarrow F_* = (f_*(x), x) \quad ,$$

where f_* is the one dimensional fixed point.

This implies that O_{F_*} lies on a smooth curve. The dynamics on the invariant Cantor set O_F is conjugate to the dynamics on O_{F_*}. The situation is thus similar to the one dimensional case, but with a twist, because rigidity does not hold:

Theorem 2: *('Non-rigidity', Carvalho, Lyubich, Martens)The conjugacy h is* at best $C^{1/2}$ *(ie: not differentiable), so that scalings do not converge to their one dimensional counterparts.*

At first glance this means that we would not see one dimensional scalings in higher dimensional phenomena, which runs counter to observations as well as numerical experiments (such as those described in Feigenbaum's talk). The attracting Cantor set $O_F \subset S_1 \cup S_1'$ lies in a very thin neighborhood of the curve $(f_*(x), x)$. Scalings can be defined as before, although now we compare lengths of 2-dimensional "matchsticks" instead of intervals.

Theorem 3: *There is $\theta \in (0,1)$ such that at the n-th level of renormalization the fraction of the 2-dimensional scalings whose values are θ^n-close to that of their 1-dimensional counterparts is at least $1-\theta^n$.*

Theorem 4: *For all $\epsilon > 0$, there is a set $X \subset O_F$ occupying a fraction of at least $1-\epsilon$ of O_F such that the conjugation $h: O_F \to O_{F_*}$ restricted to X is $C^{1+\alpha}$.*

Renormalization in Conservative Henon Maps

Folkert Tangerman, SUNY Stony Brook,

email: fmtangerman@gmail.com.

May 11, 2.45pm

The area preserving Henon map from $\mathbb{R}^2$ to itself is given by $F:(x,y) \to (a-x^2-y, x) = (f(x,y), x)$. Renormalization can be defined in a somewhat similar fashion as in the strongly dissipative case (see Martens' talk).

Start with a box S_0 such that the image $F(S_0)$ intersects S_0 in an "incomplete" horseshoe (see Figure 0.1b). One can prove that for certain a there are boxes S_1 and S_1' (They are quasi-parallelograms in the sense that their sides are smooth and close to being Euclidean segments and opposing sides are close to being translates of each other), such that $F(S_1) = S_1'$ and $F(S_1')$ maps 'across' S_1 as indicated in the second part of the figure (as an 'incomplete' horseshoe). The renormalization $\mathcal{R}F$ is defined as the (nonlinearly) rescaled version of F^2 restricted to S_1. The requirement that F be infinitely renormalizable uniquely determines the value a_0 of a.

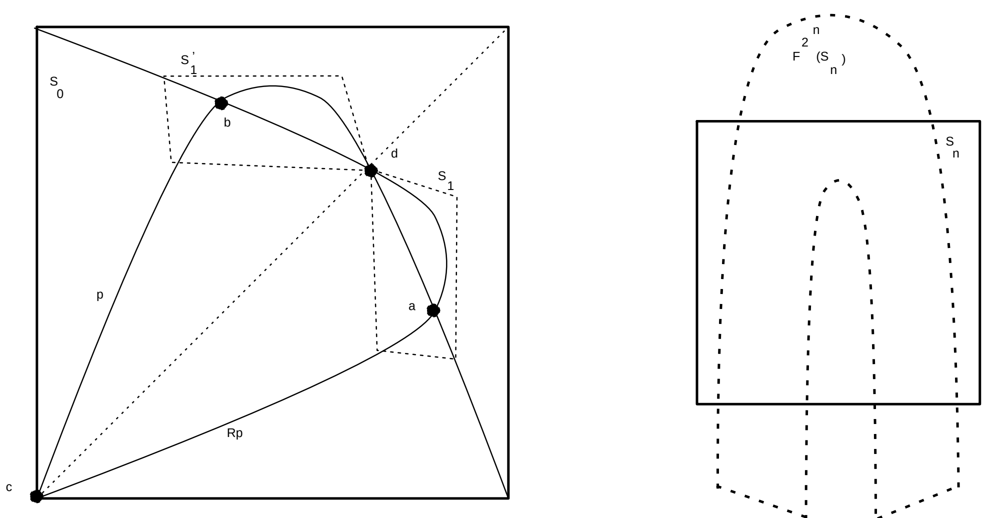

Figure 0.1: *In a) the boxes S_0 and S_1, the fixed points and points of period 2 are drawn. In b) the S_i and its image $F^{2^i}(S_i)$.*

Renormalization in the area preserving case is much less well-understood than in the strongly dissipative case. The reason is that in the latter case the image of the standard square is in a small neighborhood of a one dimensional parabola, and renormaliztion is controlled by the one-dimensional renormalization. In the former case this cannot be true because area is preserved. As an example of this, note that 'scalings' now relate domains that are *not* one-dimensional (or almost one-dimensional).

The conservative system possesses many symmetries. For all $n \geq 0$ we have the following relations:

$$(RF^n)^2 = Id \quad \text{and} \quad (RF^n)F(RF^n) = F^{-1} \quad .$$

(The relations for $n = 0$ are usually called *reversibility.*) Start by setting $a = a_0$. Denote by R the reflection in the diagonal $x = y$. The fixed point set of RF is the parabola p. The set Rp is the fixed point set of FR. Use these parabolae to define the box S_0 as in Figure 0.1 and consider F restricted to the box S_0. The two parabolae intersect in 4 points. The points c and d on the diagonal are fixed points of F, and the two off-diagonal points (denoted by a and b in the figure) form a 2-periodic orbit of F. After renormalization the rescaled $p \cap S_1$ takes the place of the diagonal, and the rescaled set $Rp \cap S_1$ takes the place of Rp. Furthermore RF plays the role of R, so that the renormalized map is also reversible (and $RF(Rp \cap S_1)$ takes the place of p). The process can now be repeated with F^2 defined on S_1.

Suppose that at each step of the renormalization certain conditions similar to so-called *a priori bounds* in one-dimensional dynamics can be verified. In that case the following results can be proved.

Theorem 1: *There is an invariant Cantor set O_F which is the limit of the orbits of period 2^n. The Lyapunov exponents along O_F are zero.*

Furthermore, $F|_{O_F}$ and $F_*|_{O_{F_*}}$ are topologically conjugate. As far as local geometry is concerned, one can prove:

Theorem 2: *The boxes S_n have "bounded geometry" (the ratios of their length and width are uniformly bounded away from 0 and infinity). The set O_F is not contained in any smooth curve.*

The Action of Automorphisms of Free Groups on Representations

Yair Minsky, Yale University,

email: yair.minsky@yale.edu.

May 11, 3.30pm

Let G be the group of Möbius transformations $z \to \dfrac{az+b}{cz+d}$. Identify these with the group $PSL(2,\mathbb{C})$ of 2 by 2 matrices (of their coefficients) with determinant 1. Elements of G act on the Riemann Sphere $\hat{\mathbb{C}}$ and, as isometries, on three dimensional hyperbolic space $\mathbb{H}^3$.

Let k be an integer and let $\mathbb{F}_k$ be the free group on k letters. Each generator is mapped to an element of G, giving an element in G^k. This gives a representation $\rho : \mathbb{F}_k \to G$. The space of such representations, modulo conjugation by elements of G, is called the character variety $\chi(\mathbb{F}_k, G)$.

Let $\mathcal{D}$ be the set of representations ρ that are faithful (i.e. injective) and whose image is a discrete, free, subgroup of G. Such a representation then acts discretely on $\mathbb{H}^3$ and the quotient $\mathbb{H}^3/\rho(\mathbb{F}_k)$ is a hyperbolic manifold or orbifold.

The following are known:

- $\mathcal{D}$ is closed.
- The interior of $\mathcal{D}$ is non-empty and connected.
- Representations in the interior of $\mathcal{D}$ correspond to Schottky groups on k generators and have a Cantor set as limit set on the Riemann Sphere.

The boundary of $\mathcal{D}$ is a complicated geometric object. The complement of $\mathcal{D}$ is a large space and contains representations that are non-faithful, or whose image is a dense subgroup of G, etc.

Automorphisms $\psi : \mathbb{F}_k \to \mathbb{F}_k$ act on representations by the rule $\rho \to \rho \circ \psi$. The set of automorphisms is a group $Aut(\mathbb{F}_k)$ and since $\mathbb{F}_k$ is free, Nielsen's method provides explicit generators for $Aut(\mathbb{F}_k)$. Inner automorphisms of $\mathbb{F}_k$, i.e. those that are defined by conjugation, act trivially on the character variety. Consider the group of outer automorphisms $Out(\mathbb{F}_k) = Aut(\mathbb{F}_k)/Inn(\mathbb{F}_k)$. This group leaves $int(\mathcal{D})$ invariant and acts discretely on $int(\mathcal{D})$.

Theorem 1: *There exists an open set $\mathcal{P}$, strictly larger than $int(\mathcal{D})$, that is invariant under $Out(\mathbb{F}_k)$ and on which $Out(\mathbb{F}_k)$ acts discretely.*

Representations in $\mathcal{P}$ are called primitive stable representations.

Definition 1: *A representation ρ is* **redundant** *if there exists a generating set $y_1, ..., y_k$ so that $\rho(y_1), ..., \rho(y_{k-1})$ generate a dense subgroup of G.*

Let $\mathcal{R}$ be the set of redundant representations. $\mathcal{R}$ is open and $\mathcal{R} \cap \mathcal{P} = \emptyset$.
Theorem 2: *[Gelander-Minsky] $Out(\mathbb{F}_k)$ is ergodic on $\mathcal{R}$.*

Anomalous Scalings in the Dissipative Henon Map

Mitchell Feigenbaum, Rockefeller University,

email: feigenb@rockefeller.edu .

May 11, 4.15pm

The existence of scalings in one-dimensional unimodal maps and their values have been well-established for decades. Their relevance that these same scalings are also observed in natural phenomena, including those described by high dimensional dissipative dynamical systems. Indeed very early on experiments by Libchaber showed period-doubling in a convective Rayleigh-Benard flow in a Helium cell. These experiment showed the correct value of the scalings (within experimental accuracy).

The 'non-rigidity' Theorem (see Theorem 2 in Martens' talk) is in apparent contradiction with these observations. The resolution of this contradiction is that most scalings are in fact very close to their one-dimensional counterparts. There *are* indeed anomalous scalings whose values are far from the one-dimensional ones. With increasing level n these scalings become exponentially (in n) rare.

Let $\nu_n(x)$ be the fraction of level n scalings in the dissipative Henon map, whose discrepancy with their one-dimensional counterparts is greater or equal to x. Detailed numerical experiments suggest that (see Figure 0.2) there is a number $\theta \in (0, 1)$ such that (asymptotically in n)

$$\nu_{n+1}(x) = \nu_n(\theta^{-1}x) \quad .$$

Theorem 3 in Martens' talk is consistent with these data.

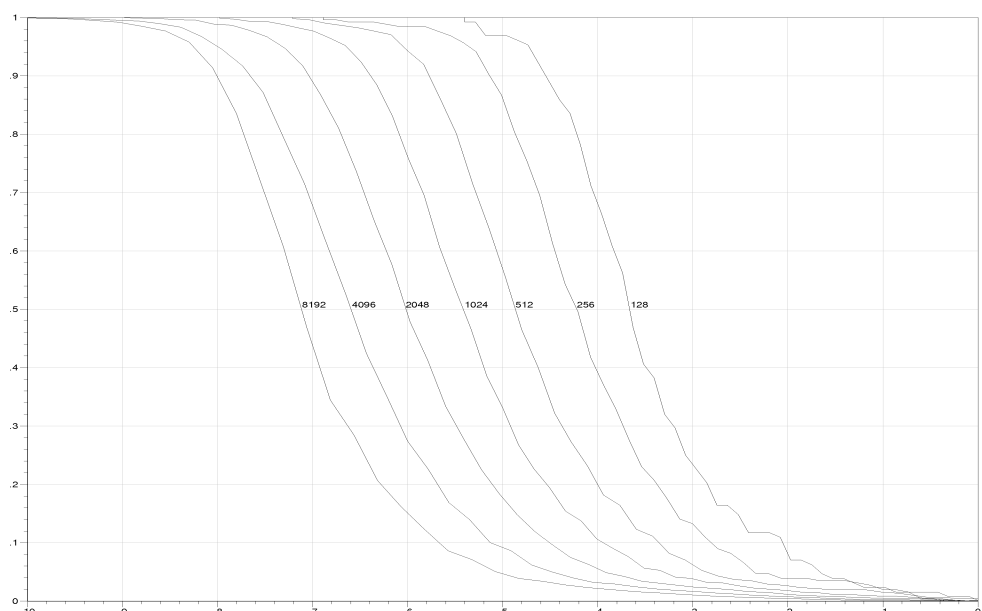

Figure 0.2: *The curves are the distribution functions for scalings in error $> 10^{-n}$, $-n$ the abscissa, for intervals of the labeled number of iterations. (Magnify to obtain a more accurate rendering).*